ECE 580 — **SPRING 2019**

Professor Tamer Başar — **May 3, 2019**

FINAL EXAM

DUE DATE & TIME : Tuesday, May 7, 2019; 11:00 am.
Since I will be out of town, leave your final exam with my secretary, Angie Ellis, next door to my office at CSL, or in case she is not around, slide under my door (and send me an email to that effect). You can also send your final exam directly to me by email, with "580 Final" in the subject line.

Answer the four questions below, starting each one on a separate sheet.

Problem 1 (25 points)

Let $C[-1,1]$ denote the normed vector space of continuous functions on $[-1,1]$, equipped with the standard maximum norm. We wish to find a linear functional f on $X = C[-1,1]$, with minimum norm, that satisfies the side conditions:

$$f(x_1) = 1\,, \quad f(x_2) = 2$$

where

$$x_1(t) = 2t^2 - t\,, \quad x_2(t) = -t^2 + t + 1$$

i) Is this a feasible problem? Justify your answer.

ii) If your answer to part (i) is in the affirmative, obtain the solution.

iii) Repeat (i) and (ii) above with $X = L_2[-1,1]$, instead of $C[-1,1]$, where $L_2[-1,1]$ is the standard Hilbert space of square-integrable functions on $[-1,1]$.

iv) Would the result in (iii) be any different if X is the normed vector space of continuous functions on $[-1,1]$ with the L_2 norm?

v) Let x^* be the solution you obtained in part (ii). Describe the set of all $x \in C[-1,1]$ that are orthogonal to x^*, that is $< x, x^* >= 0$.

Problem 2 (25 points)

Let $L_2[0,1]$ be the standard Hilbert space of square-integrable functions on $[0,1]$. You are given the functional $J : L_2[0,1] \to \mathbf{R}$:

$$J(x) = \int_0^1 [n(t) + x(t)][m(t) + (\mathcal{L}x)(t)]\,dt\,,$$

where $m, n \in L_2[0,1]$, and $\mathcal{L} : L_2[0,1] \to L_2[0,1]$ is a linear operator given by

$$(\mathcal{L}x)(t) = \int_0^t k(t,s)x(s)\,ds\,,$$

with $k \in L_2[[0,1] \times [0,1]]$.

i) Obtain the Frechét derivative of J.
[Make sure that you properly identify the space where the derivative belongs, and also verify that what you obtained is indeed the Frechét derivative.]

ii) Repeat (i) for the case when $L_2[0,1]$ is replaced throughout by $C[0,1]$, with the standard maximum norm.

iii) Noting that $J(x)$ can be written as

$$J(x) = (n,m) + (n,\mathcal{L}x) + (m,x) + (x,\mathcal{L}x)\,,$$

where $(\cdot,\cdot)$ is the standard inner product on $L_2[0,1]$, obtain the conditions (if any) on n, m, k for J to admit a unique global minimum in $L_2[0,1]$.

iv) When the conditions in (iii) (if any) are satisfied, obtain the minimizing solution in terms of the linear operator $\mathcal{L}$, its adjoint $\mathcal{L}^*$, n and m. Also, obtain an expression for $\mathcal{L}^*$.

v) Apply the result of (iv) above to the specific problem where

$$k(t,s) = 1 + s\,, \quad n(t) \equiv 0\,, \quad m(t) = \frac{5}{6}t^3 - t^2 - \frac{1}{2}t - \frac{1}{2}$$

Problem 3 (25 points)

Consider the calculus of variations problem of minimizing

$$J(x,y) = \int_0^1 f[t, x(t), \dot{x}(t), y(t), \dot{y}(t)]\,dt$$

over $(x,y) \in (D[0,1] \times D[0,1])$, subject to the constraints

$$x(0) + y(0) = 1\,, \quad \int_0^1 g_i[t, x(t), y(t)]\,dt = 0, \quad i = 1, 2.$$

Here f and g_i are continuously differentiable in t, x and y, and f is twice continuously differentiable in $\dot{x}$ and $\dot{y}$. Furthermore, $D[0,1]$ stands for the class of continuously differentiable functions on $[0,1]$, equipped with the maximum norm (involving both the function and its derivative).

i) Let (x_o, y_o) provide a local (minimizing) solution to this problem. What additional conditions (if any) need to be satisfied for (x_o, y_o) to be a *regular point* of the given constraints.

ii) Assuming that (x_o, y_o) is a regular point, obtain a **complete** set of first-order necessary conditions that it should satisfy (for it to be a minimizing solution).
[Note that $x(1)$ and $y(1)$ are free end points.]

Problem 4 (25 points)

Let H be a real Hilbert space, and p, q be two elements in H with

$$\|p\| = \|q\| = 2\,, \quad (p,q) = -1$$

where $(\cdot,\cdot)$ denotes the inner product. Consider the (primal) problem of minimizing

$$J(x) = (p, x)$$

over $x \in H$, subject to the inequality constraints:

$$(x,x) \leq \frac{17}{2}\,, \quad (q,x) \leq \frac{1}{2}$$

i) Formulate the **dual** problem and obtain its solution.

ii) What is the (minimum) value of the primal problem?

iii) Let us now replace the second inequality constraint above by

$$(q,x) \leq c$$

where c is a positive scalar. Denote the minimum value of the corresponding primal problem by $J_o(c)$. Compute

$$\frac{dJ_o(c)}{dc} \quad \text{at } c = 0.5\,,$$

without obtaining an explicit expresion for $J_o(c)$.
[Explain and justify your reasoning (see section 8.5 of the Text).]

iv) In the problem formulation, now let $H = L_2[-1,1]$, the standard Hilbert space of square-integrable functions on $[-1,1]$, and

$$p(t) = \sqrt{6}t\,, \qquad q(t) = -\frac{\sqrt{6}}{4}t + r(t)\,,$$

where $r(t)$ is picked such that $\|q\| = 2$ and $(p, q) = -1$.
First obtain $r(t)$, and then an explicit expression for the solution to the primal problem. Denoting the solution you obtained here by x^o, check to see whether $J(x^o)$ equals the minimum value you computed in part (ii).

GOOD LUCK

and

HAVE A PLEASANT SUMMER